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STABILITY ANALYSIS OF STAGE-STRUCTURED PREY MODEL INCORPORATING A PREY REFUGE

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Abstract

In this paper, a food chain model consisting of mid- predator and top predator-stage structured prey involving two functional response Lotka-Voltera type and Holling type-II with a prey refuge, is proposed and analyzed. For this model, it is assumed that the prey growth logistically in the absence of predator. The role of prey refuges in predator-prey model is investigated. The existence, uniqueness and boundedness of the solution of this model are studied. The stability analysis of all possible equilibrium points are studied.Suitable Lyapunov functions are used to study the global dynamics of the proposed model.Finally, numerical simulation for different sets of parameter value and for different sets of initial conditions are carried out to investigate the influence of parameters on the dynamical behavior of the model and to support our obtained analytical results of the model. Keywords: food chain, Lyapunov function, refuge, stability analysis, stage-structure.

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I. INTRODUCTION

The dynamical behavior of ecological system involving prey-predator interaction or competition interaction or mutualism interaction has long been and will continue to be one of the dominant topics in both applied mathematics and theoretical ecology due to its universal existence and importance [1].The prey-predator system exhibits age and stage based dynamics that has been widely studied using mathematical models [2]. The age factor is importance for the dynamics and evolution of many mammals. The rate of survival, growth and reproduction almost depend on age or development stage and it has been noticed that the life history of many species is composed of at least two stages, immature and mature, and the species in the first stage may often neither interact with other species nor reproduce, being raised by their mature parents. Stage structure models have been investigated extensively in the recent decades. The single species model with stage structure was studied by Aiello and Freedman (1990). Most of classical prey-predator models of two species in the literature assumed that all predators are able to attack their prey and reproduce ignoring the fact that the life cycle of most animals consists of at least two stages (immature and mature). Two species prey-predator models with stage structure were investigated by many researchers see for example (Wang and Chen, 1997), (Magnasson, 1999) and the references therein. However, the stage structure prey-predator models with or without time delays have been investigated by Gourley and Kuang (2004) and recently by Kar and Matsuda (2006) [3,4,5]. Naji, R.K., et al. [6] studied the stability of a stage structure prey-predator model with predator-dependent functional response. Recently, autonomous systems with a stage structure have been considered in [7,8,9,10]. A predator-prey system with stage structure for the prey is studied by Cui and Takeuchi [11], they provided a sufficient and necessary condition to guarantee permanence of the system. A three species Lotka-Voltera type food chain model with stage structure and time delays is investigated by Xu et al. [12], they assumed that the individuals in each species may belong to immature or mature class.Raid Kamel Naji and Alla Tariq Balasim (2007) studied dynamical behavior of a three species food chain model with Beddington-DeAngelis functional response.On the other hand some of the empirical and theoretical work have investigated the effect of prey refuges and drawn a conclusion that the refuges used by prey have a stabilizing effect on the considered interactions and prey extinction can be prevented by the addition of refuges [13,14]. Z.Jawad Kadhim , Azhar Abbas Majeed and R. Kamel Naji in [15] studied the stability analysis of food web stage structured prey-predator model with refuge involving Lotka-Volterra type of function response. In this paper, the food chain stage structured prey-predator model involving prey refuge with two different functional respones is proposed and analyzed, so that the prey growing logistically in the absence of predator. The effect of prey refuge and prey stage structure on the dynamical behavior of the food chain model are investigated theoretically.

II. THE MATHEMATICAL MODEL

Consider the food chain model consisting of mid -predator and top predator stage-structure prey in which the prey species growth logistically in the absence of predation, while the predators decay exponentially in the absence of prey species. It is assumed that the prey population divides into two compartments: immature prey population W_1(t) that represents the population size at time t and mature prey population W_2(t) which denotes to population size at time t. Furthermore the population size of the mid- predator at time t is denoted by W_3(t), while W_4(t) represents the population size of top predator at time t . Now in order to formulate the dynamics of such system the following assumptions are considered:

1. The immature prey depends completely in its feeding on the mature prey that growth logistically with intrinsic growth rate $\alpha > 0$ and carrying capacity $k > 0$. The immature prey individuals grown up and becomes mature prey individuals with grown up rate $\beta > 0$. However the mature prey facing death with natural death rate $d_1 > 0$.

2. There is type of protection of the prey species from facing predation by mid- predators with refuge rate constant $m \in (0, 1)$.

3. The mid-predator consumed the mature prey individual only according to the Lotka-Voltera type of functional response with predation rate $c_1 > 0$ and the top predator consumed the mid- predator individual only according to Holling type-II functional response with predation rate $c_2 > 0$ and half saturation constant $b > 0$ respectively and contribute a portion of such food with conversion rates $0 < e_1 < 1$ *and* $0 < e_2 < 1$ respectively. Finally, in the absence of food the mid- predator and the top predator facing death with natural death rate $d_2 > 0$ *and* $d_3 > 0$.

Therefore the dynamics of the above proposed model can be represented by the following set of first order nonlinear differential equations .

$$\frac{dW_1}{dT} = \alpha W_2 \left(1 - \frac{W_2}{k}\right) - \beta W_1$$

$$\frac{dW_2}{dT} = \beta W_1 - d_1 W_2 - c_1 (1-m) W_2 W_3 \qquad (1)$$

$$\frac{dW_3}{dT} = e_1 c_1 (1-m) W_2 W_3 - d_2 W_3 - \frac{c_2 W_3}{b + W_3} W_4$$

$$\frac{dW_4}{dT} = \frac{e_2 c_2 W_3}{b + W_3} W_4 - d_3 W_4$$

with initial conditions $W_i(0) \geq 0$, $i = 1, 2, 3, 4$. Not that the above proposed model has twelve parameters in all which make the analysis difficult. So in order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters :

$$t = \alpha \mathrm{T},\ a_1 = \frac{\beta}{\alpha},\ a_2 = \frac{d_1}{\alpha},\ a_3 = \frac{d_2}{\alpha},\ a_4 = \frac{e_1 c_1 k}{\alpha}, a_5 = \frac{c_2}{\alpha},\ a_6 = \frac{b c_1}{\alpha},$$

$$a_7 = \frac{e_2 c_2}{\alpha},\ a_8 = \frac{d_3}{\alpha},\ w_1 = \frac{W_1}{k},\ w_2 = \frac{W_2}{k},\ w_3 = \frac{c_1 W_3}{\alpha},\ w_4 = \frac{c_1 W_4}{\alpha}.$$

Then the non-dimensional from of system (1) can be written as:

$$\frac{dw_1}{dt} = w_1 \left[\frac{w_2(1-w_2)}{w_1} - a_1\right] = w_1 f_1(w_1, w_2, w_3, w_4)$$

$$\frac{dw_2}{dt} = w_2 \left[\frac{a_1 w_1}{w_2} - a_2 - (1-m) w_3\right] = w_2 f_2(w_1, w_2, w_3, w_4) \qquad (2)$$

$$\frac{dw_3}{dt} = w_3 \left[a_4(1-m) w_2 - a_3 - \frac{a_5 w_4}{a_6 + w_3}\right] = w_3 f_3(w_1, w_2, w_3, w_4)$$

$$\frac{dw_4}{dt} = w_4 \left[\frac{a_7 w_3}{a_6 + w_3} - a_8\right] = w_4\ f_4(w_1, w_2, w_3, w_4)$$

with $w_1(0) \geq 0, w_2(0) \geq 0, w_3(0) \geq 0$ *and* $w_4(0) \geq 0$. It is observed that the number of parameters have been reduced from twelve in the system (1) to nine in the system (2). Obviously the interaction functions of the system (2) are continuous and have continuous partial derivatives on the following positive four dimensional space.

$$R_+^4 = \left\{ \begin{matrix} (w_1, w_2, w_3, w_4) \in R^4 : w_1(0) \geq 0, w_2(0) \geq \\ 0, w_3(0) \geq 0\ w_4(0) \geq 0. \end{matrix} \right\}$$

Therefore these functions are Lipschitzian on R_+^4, and hence the solution of the system (2) exists and is unique. Further, all the solutions of system (2) with non-negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem **(2.1)**: All the solutions of system (2) which initiate in R_+^4 are uniformly bounded.

Proof: Let $(w_1(t), w_2(t), w_3(t), w_4(t))$ be any solution of the system (2) with non-negative initial condition $(w_{10}, w_{20}, w_{30}, w_{40}) \in R_+^4$. Now according to the first equation of system (2) we have

$$\frac{dw_1}{dt} = w_2(1 - w_2) - a_1 w_1$$

So, $\frac{dw_1}{dt} \leq \frac{1}{4} - a_1 w_1$.

Now, by using the comparison theorem on the above differential inequality with the initial point $w_1(0) = w_{10}$ we get:

$$w_1(t) \leq \frac{1}{4a_1} + \left(w_{10} - \frac{1}{4a_1}\right) e^{-a_1 t}$$

Thus, $\lim_{t\to\infty} w_1(t) \leq \frac{1}{4a_1}$. So, $\lim_{t\to\infty} w_1(t) \leq \frac{1}{4a_1}$

Now define the function:

$M(t) = w_1(t) + \frac{a_4}{a_5} w_2(t) + \frac{1}{a_5} w_3(t) + \frac{1}{a_7} w_4(t)$
and then taken the time derivative of $M(t)$ along the solution of the system (2) we get:

$$\frac{dM}{dt} \leq \frac{1}{4} + \frac{2a_1 a_4}{a_5} - s M, \ \textit{where}\ \ s = \min\left\{\frac{a_1 a_4}{a_8}, a_2, a_3, a_8\right\}.$$

$$\textit{Then}\ \ \frac{dM}{dt} + sM \leq F, \quad \textit{where}\ \ F = \frac{1}{4} + \frac{a_4}{2a_5}.$$

Again by solving this differential inequality for the initial value $M(0) = M_0$, we get**:**

$$M(t) \leq \frac{F}{s} + \left(M_0 - \frac{F}{s}\right) e^{-st}$$

$$\textit{Then}\quad \lim_{t\to\infty} M(t) \leq \frac{F}{s}$$

So, $0 \leq M(t) \leq \frac{F}{s}$, $\forall\, t > 0$. Hence all the solutions of system (2) are uniformly bounded and the proof is complete.

III. THE EXISTENCE OF EQUILIBRIUM POINTS

In this section, the existence of all possible equilibrium points of system (2) is discussed. It is observed that, system (2.2) has at most four equilibrium points, which are mentioned in the following:

■ The equilibrium point $E_0 = (0, 0, 0, 0)$ which known as the vanishing point is always exists.

■ The first equilibrium point $E_1 = (\bar{w}_1, \bar{w}_2, 0, 0)$ exists uniquely in $Int.R_+^2$ (Interior of R_+^2) of $w_1 w_2 - plane$ if there is a positive solution to the following set of equations:

$$\frac{w_2(1 - w_2)}{w_1} - a_1 = 0 \qquad (3\,a)$$

$$\frac{a_1 w_1}{w_2} - a_2 = 0 \qquad (3\,b)$$

From equation (2.3 a) we have,

$$w_2(1 - w_2) = a_1 w_1. \qquad (3\,c)$$

Now by substituting equation (3 c) in equation (3 b) we obtain that:

$\bar{w}_2 = 1 - a_2$, and $\bar{w}_1 = \frac{a_2}{a_1}(1 - a_2)$.

Therefore, $\bar{w}_2 > 0$ and $\bar{w}_1 > 0$

provided that $a_2 < 1$ $(3\,d)$

■ The three species equilibrium point $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ exists uniquely in the $Int.R^3_+$ of $w_1 w_2 w_3 - space$ if there is a positive solution to the following set of equations:

$$w_2\,(1 - w_2) = a_1\, w_1 \qquad (4\,a)$$

$$\frac{a_1\, w_1}{w_2} - a_2 - (1 - m) w_3 = 0 \qquad (4\,b)$$

$$-a_3 + a_4\,(1 - m)\, w_2 = 0 \qquad (4\,c)$$

From equation (2.4 c) we have,

$$\breve{w}_2 = \frac{a_3}{a_4\,(1 - m)} \qquad (4\,d)$$

Substituting equation (2.4 d) in equation (2.4 a) we get:

$$\breve{w}_1 = \frac{a_3}{a_1\, a_4\,(1 - m)} \left[\frac{a_4\,(1 - m) - a_3}{a_4\,(1 - m)} \right] \qquad (4\,e)$$

Now, by replacement equations (2.4 e) and (2.4 d) in equation (2.4 b) we obtain:

$$\breve{w}_3 = \frac{a_4\,(1 - m)\,(1 - a_2) - a_3}{a_4\,(1 - m)^2} \qquad (4\,f)$$

Consequently, the first three species equilibrium point $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ exists uniquely in the $Int.R^3_+$ of $w_1 w_2 w_3 - space$ provided that in addition of condition (3 d) and the following condition holds:

$$a_3 < \min\,\{ a_4\,(1 - m)\,,\, a_4\,(1 - m)\,(1 - a_2)\}. \qquad (4\,g)$$

■ Finally, the positive equilibrium point $E_4 = (w_1^*, w_2^*, w_3^*, w_4^*)$ exists where if there is a positive solution to the following set of equations:

$$w_2\,(1 - w_2) = a_1\, w_1 \qquad (5\,a)$$

$$\frac{a_1\, w_1}{w_2} - a_2 - (1 - m)\, w_3 = 0 \qquad (5\,b)$$

$$-a_3 + a_4\,(1 - m)\, w_2 - \frac{a_5\, w_4}{a_6 + w_3} = 0 \qquad (5\,c)$$

$$-a_8 + \frac{a_7\, w_3}{a_6 + w_3} = 0 \qquad (5\,d)$$

From equation (5 d)we obtain :

$$w_3^* = \frac{a_6\;a_8}{a_7 - a_8} \qquad (5\,e)$$

Substituting equation (5 e) and (5 a)in equation (5 b) we obtain that:

$$w_2^* = 1 - a_2 - \left[\frac{(1 - m)\, a_6\, a_8}{a_7 - a_8} \right] \qquad (5\,f)$$

replacement equation (5 f) in equation (5 a) we obtain that:

$$w_1^* = \frac{{w_2}^*}{a_1}(1 - w_2^*) \qquad (5\,g)$$

Now ,by substituting equation (5 f) and (5 e) in equation (5 d) we obtain that:

$$w_4^* = \frac{1}{a_5}\left(a_6 + \frac{a_6\, a_8}{a_7 - a_8}\right)[\, a_4(1 - m) w_2^* - a_3\,] \quad (5\,k)$$

Consequently, the positive equilibrium point $E_3 = (w_1^*, w_2^*, w_3^*, w_4^*)$ exists uniquely in the $.R^4_+$, provided that the following condition holds:

$$a_8 < a_7 \qquad (5\,h)$$

$$\frac{a_3}{a_4(1 - m)} < w_2^* < 1 \qquad (5\,i)$$

$$a_2 + (1 - m)\frac{a_6\, a_8}{a_7 - a_8} < 1 \qquad (5\,q)$$

IV. THE LOCAL STABILITY ANALYSIS OF SYSTEM

In this section, the local stability analysis of system (2) around each of the above equilibrium points are discussed through computing the Jacobian matrix $J(w_1, w_2, w_3, w_4)$ of system (2) at each of them which given by:

$$J = [\,u_{ij}\,]_{4\times 4} \quad , \quad where \qquad (6)$$

$$u_{11} = -a_1\ , u_{12} = 1 - 2\,w_2\,, u_{13} = 0\ , u_{14} = 0\ , u_{21} = a_1\,, u_{22} = -a_2 - (1-m)w_3\,,$$

$$u_{23} = -(1-m)w_2\,, u_{24} = 0, u_{31} = 0\,, u_{32} = a_4\,(1-m)w_3,$$

$$u_{33} = -a_3 + a_4\,(1-m)\,w_2 - \frac{a_5\,a_6 w_4}{(a_6+w_3)^2}\,, u_{34} = -\frac{a_5\,w_3}{a_6+w_3}\,, u_{41} = 0, u_{42} = 0,$$

$$u_{43} = \frac{a_6\,a_7 w_4}{(a_6+w_3)^2}\,,\ \ u_{44} = -\,a_8 + \frac{a_7\,w_3}{a_6+w_3}$$

The local stability analysis at E_0

The Jacobian matrix of system (2) at E_0 can be written as:

$$J(E_0) = \begin{bmatrix} -a_1 & 1 & 0 & 0 \\ a_1 & -a_2 & 0 & 0 \\ 0 & 0 & -a_3 & 0 \\ 0 & 0 & 0 & -a_8 \end{bmatrix} \qquad (7\,a)$$

Then the characteristic equation of $J(E_0)$ *is given by:*

$$[\,\lambda^2 + A_1\,\lambda + A_2\,]\,(\,-a_3 - \lambda\,)\,(\,-a_8 - \lambda\,) = 0\ , \qquad (7\,b)$$

where $A_1 = a_1 + a_2$, $A_2 = a_1\,(\,a_2 - 1\,)$.

So, either

$$(\,-a_3 - \lambda\,)\,(-a_8 - \lambda\,) = 0\ , \qquad (7\,c)$$

which gives two of the eigenvalues of $J(E_0)$ by:

$$\lambda_{0w_3} = -a_3 < 0 \ \text{ and } \ \lambda_{0w_4} = -a_8 < 0$$

or $\lambda^2 + A_1\lambda + A_2 = 0$, $\qquad (7\,d)$

which gives the other two eigenvalues of $J(E_0)$ by:

$$\lambda_{0w_1} = \frac{-A_1}{2} + \frac{1}{2}\sqrt{{A_1}^2 - 4\,A_2}\ ,\ \lambda_{0w_2} = \frac{-A_1}{2} - \frac{1}{2}\sqrt{{A_1}^2 - 4\,A_2}\ .$$

Therefore, if the following condition holds:

$$a_2 > 1 \qquad (7\,e)$$

E_0 is locally asymptotically stable in the R_+^4 . However, it is a saddle point otherwise.

The local stability analysis at E_1

The Jacobian matrix of system (2) at E_1 can be written as:

$$J(E_1) = \begin{bmatrix} -a_1 & 1 - 2\overline{w}_2 & 0 & 0 \\ a_1 & -a_2 & -(1-m)\overline{w}_2 & 0 \\ 0 & 0 & -a_3 + a_4(1-m)\overline{w}_2 & 0 \\ 0 & 0 & 0 & -a_8 \end{bmatrix} \qquad (8a)$$

Then the characteristic equation of $J(E_1)$ is given by:

$$[\,\lambda^2 + \bar{A}_1\lambda + \bar{A}_2\,]\,(-a_3 + a_4(1-m)\overline{w}_2 - \lambda\,)\,(-a_8 - \lambda\,) = 0\,, \qquad (8\,b)$$

where $\bar{A}_1 = a_1 + a_2\,,$ $\bar{A}_2 = a_1\,(1 - a_2\,)$.

So, either

$$(-a_3 + a_4\,(1-m)\overline{w}_2 - \lambda\,)\,(-a_8 - \lambda\,) = 0\ , \qquad (8\,c)$$

which gives two of the eigenvalues of $J(E_1)$ by:

$$\lambda_{1w_3} = -a_3 + a_4\,(1-m)\,\overline{w}_2, \ \text{ and } \ \lambda_{1w_4} = -a_8 < 0$$

Or $\lambda^2 + \bar{A}_1\,\lambda + \bar{A}_2 = 0$, $\qquad (8\,d)$

which gives the other two eigenvalues of $J(E_1)$ by:

$$\lambda_{1w_1} = -\frac{\bar{A}_1}{2} + \frac{1}{2}\sqrt{{\bar{A}_1}^2 - 4\,\bar{A}_2}\ ,\ \lambda_{1w_2} = -\frac{\bar{A}_1}{2} - \frac{1}{2}\sqrt{{\bar{A}_1}^2 - 4\,\bar{A}_2}\ .$$

Therefore, if the following condition holds:

$$\overline{w}_2 < \frac{a_3}{a_4(1-m)} \qquad (8\,e)$$

In addition of condition (3 d) we obtain that E_1 is locally asymptotically stable in the R_+^4 . However, it is a saddle point otherwise.

The local stability analysis at E_2

The Jacobian matrix of system (2) at E_2 can be written as:

$$J(E_2) = \left[\, b_{ij} \,\right]_{4\times4} \quad , \qquad (9\,a)$$

Where,

$$b_{11} = -a_1 < 0 \;, b_{12} = 1 - 2\,\breve{w}_2 \;, b_{13} = 0 \;\;, b_{14} = 0 \;, b_{21} = a_1 > 0 \,, b_{21} = a_1 > 0 \,,$$

$$b_{22} = -a_2 - (1-m)\,\breve{w}_3 \; < 0 \;, b_{23} = -(1-m)\,\breve{w}_2 < 0, b_{24} = 0 \;, b_{31} = 0 \;,$$

$$b_{32} = a_4\,(1-m)\,\breve{w}_3 \; > 0, b_{33} = -a_3 + a_4\,(1-m)\,\breve{w}_2 \;\;, b_{34} = -\frac{a_5\;\breve{w}_3}{a_6 + \;\breve{w}_3} < 0$$

$$b_{41} = 0 \;, b_{42} = 0 \;, b_{43} = 0 \;, b_{44} = -a_8 + \frac{a_7\;\breve{w}_3}{a_6 + \;\breve{w}_3}.$$

Then the characteristic equation of $J(E_2)$ is given by:

$$[\lambda^3 + \; R_1\lambda^2 + R_2\,\lambda + R_3]\left(-a_8 + \frac{a_7\;\breve{w}_3}{a_6 + \;\breve{w}_3} - \lambda\right) = 0 \qquad (9\,b)$$

where

$$R_1 = \; -(\,b_{11} + b_{22}) > 0 \,,$$

$$R_2 = -(b_{11}b_{22} + b_{23}b_{32} + b_{12}b_{21}),$$

$$R_3 = b_{11}b_{23}b_{32} > 0,$$

$$So, either \qquad -a_8 + \frac{a_7\;\breve{w}_3}{a_6 + \;\breve{w}_3} - \lambda = 0 \;\;. \;\;(9\,c)$$

Or $\quad \lambda^3 + \; R_1\lambda^2 + R_2\,\lambda + R_3 = 0 \;\;. \qquad (9\,d)$

Hence from equation (9 c) we obtain that:

$\lambda_{2w_3} = -a_8 + \frac{a_7\;\breve{w}_3}{a_6 + \breve{w}_3}$ which is negative if inaddition of condition (5 h) the following condition hold:

$$\breve{w}_3 < \frac{a_6 a_8}{a_7 - a_8} \qquad (9\,e)$$

On the other hand by using Routh-Hawirtiz criterion equation (9 d) has roots (eigenvalues) with negative real parts if and only if

$$R_1 > 0 \;, R_3 > 0 \quad and \quad \Delta = R_1\,R_2 - R_3 > 0$$

Straightforword computation shows that $\Delta > 0$

$$\text{provided that} \;\; \frac{a_3}{a_4(1-m)} > \frac{1}{2} \qquad (9\,f)$$

Now it is easy to verify that $R_1 > 0$ and $R_3 > 0$ under condition (4 g).Then all the eigenvalues λ_{2w_1} , λ_{2w_2} and λ_{2w_3} of equation (9 d) have negative real parts. So, E_2 is locally asymptotically stable .

The local stability analysis at E_3

The Jacobian matrix of system (2) at E_3 can be written as:

$$J = \left[\, d_{ij} \,\right]_{4\times4} \quad , \; where \quad (10\,a)$$

$$d_{11} = -a_1 \;\;, d_{12} = 1 - 2\,w_2^* \,, d_{13} = 0 \;\;, d_{14} = 0 \;, d_{21} = a_1, d_{22} = -a_2 - (1-m)\,w_3^*$$

$$d_{23} = -(1-m)\,w_2^* \;\;, d_{24} = 0 \;\;, d_{31} = 0, d_{32} = a_4\,(1-m)\,w_3^*$$

$$d_{33} = -a_3 + a_4\,(1-m)w_2^* - \frac{a_5 a_6 w_2^*}{(a_6 + w_3^*)^2}, d_{34} = \frac{-a_5 w_3^*}{a_6 + w_3^*} \,, \;\; d_{41} = 0,$$

$$d_{42} = 0, \;\; d_{43} = \frac{a_6 a_7 w_4^*}{(a_6 + w_3^*)^2} \,, d_{44} = -a_8 + \frac{a_7 w_3^*}{a_6 + w_3^*}.$$

Then the characteristic equation of $J(E_3)$ is given by:

$$[\lambda^4 + C_1\,\lambda^3 + C_2\,\lambda^2 + C_3\,\lambda + C_4\,] = 0 \quad (10\,b)$$

$$where \quad C_1 = \mu_1 \;\,, \; C_2 = \mu_2 + \mu_3 \;\,, \; C_3 = \mu_4 + \mu_5 \;, \; C_4 = \mu_6 \;\;,$$

and $\quad \mu_1 = -(\,\Gamma_1 + d_{33}), \;\; \mu_2 = \Gamma_2 - \Gamma_3 \,, \;\; \mu_3 = d_{33}\Gamma_1$

$$\mu_4 = d_{34}d_{43}\Gamma_2 + d_{11}d_{23}d_{32}\ ,\ \mu_5 = -d_{33}\Gamma_2\ ,\mu_6 = -d_{34}d_{43}\ \Gamma_2$$

$$with \quad \Gamma_1 = d_{11} + d_{22}\ ,\ \Gamma_2 = d_{11}d_{22} - d_{12}d_{21}\ ,\ \Gamma_3 = d_{34}d_{43} - d_{23}d_{32}\ ,$$

Consequently: $\Delta_1 = C_1\ C_2 - C_3 = \mu_1(\ \mu_2 + \ \mu_3) - (\ \mu_4 + \mu_5)$ Now we have $C_i > 0$, $i = 1, 3, 4$ and $\Delta_1 > 0$ if and only if the following conditions are hold:

$$\mu_1\mu_2 - \mu_5 > \mu_4 - \mu_1\mu_3 \quad (\ 10\ c\ )$$

and $\quad \Delta_2 = C_3\ (\ C_1C_2 - C_3\ ) - C_1^2\ C_4$

$\Delta_2 > 0$ if and only if the following conditions are hold:

$$-\Gamma_1 > d_{33} \quad (\ 10\ d\ )$$

$$\mu_4 > d_{33}\Gamma_2 \quad (\ 10\ e\ )$$

$$[\mu_1\ (\ \mu_2 + \mu_3) - (\mu_4 + \mu_5)](\mu_4 + \mu_5) > \mu_1\mu_6 \quad (\ 10\ f\ )$$

So, according to Routh-Hawirtiz criterion the equilibrium point E_4 is locally asymptotically stable.

V. THE GLOBAL STABILITY ANALYSIS OF SYSTEM

In this section the global stability analysis for the equilibrium points, which are locally asymptotically stable, of system (2) is studied analytically with the help of Lyapunov method as shown in the following theorems.

***Theorem* (2)**: Assume that the equilibrium point $E_0 = (\ 0\ ,0\ ,0\ ,0\ )$ of system (2) is locally asymptotically stable in the R_+^4 . Then the equilibrium point E_0 is globally asymptotically stable.

Proof: Consider the following function:

$$V_1(\ x\ ,y\ ,z\ ,w\ ) = c_1\ w_1 + c_2\ w_2 + c_3\ w_3 + c_4\ w_4$$

where $c_1\ ,c_2\ ,c_3\ and\ c_4$ are positive constants to be determine .

Clearly $V_1: R_+^4 \rightarrow R$ is a C^1 positive definite function. Now by differentiating V_1 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_1}{dt} = a_1(-c_1 + c_2)\ w_1 + [\ c_1(1 - w_2) - c_2\ a_2\ ]\ w_2 + (c_3\ a_4 - c_2)(1 - m)w_2w_3 + (c_4\ a_7 - c_3a_5)\frac{w_3w_4}{a_6 + w_3} - c_3a_4w_3 - c_4a_8w_4.$$

By choosing $c_1 = c_2 = 1\ ,\ c_3 = \frac{1}{a_4}\ ,\ c_4 = \frac{a_5}{a_4}$ we get:

$\frac{dV_1}{dt} < -(\ a_2 - 1\ )\ w_2$ Now, under the condition (7 e) we get that $\frac{dV_1}{dt}$ is negative definite and hence V_1 is strictly Lyapunov function. Thus we obtain that E_0 is globally asymptotically stable and the proof is complete.

***Theorem* (3)**: Assume that the equilibrium point $E_1 = (\ \overline{w}_1\ ,\overline{w}_2\ ,0\ ,0\ )$ of system (2) is locally asymptotically stable in the R_+^4 , Then E_1 is globally asymptotically stable on any region $\Omega_1 \subset R_+^4$ that satisfies the following conditions:

$$\frac{1}{w_1} - \frac{(w_2 + \overline{w}_2)}{\overline{w}_1} + \frac{a_1}{(1 - m)\ w_2\ \overline{w}_2} \leq 2\sqrt{\frac{a_1}{(1 - m)\ w_1\ w_2\ \overline{w}_2}} \quad (\ 11\ a\ )$$

$$\frac{w_2 + \overline{w}_2}{\overline{w}_1} < \frac{1}{w_1} + \frac{a_1}{(1 - m)\ w_2\ \overline{w}_2} \quad (\ 11\ b\ )$$

$$w_2^2 < \overline{w}_2 \quad (\ 11\ c\ )$$

Proof: Consider the following function:

$$V_2(\ w_1, w_2\ ,w_3\ ,w_4\ ) = c_1\left(w_1 - \overline{w}_1 - \overline{w}_1\ln\frac{w_1}{\overline{w}_1}\right) + c_2\left(w_2 - \overline{w}_2 - \overline{w}_2\ \ln\frac{w_2}{\overline{w}_2}\right) + c_3w_3 + c_4w_4$$

where $c_1\ ,c_2\ ,c_3\ and\ c_4$ are positive constants to be determine .

Clearly $V_2: R_+^4 \rightarrow R$ is a C^1 positive definite function. Now by differentiating V_2 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_2}{dt} = -c_1\,\frac{(\bar{w}_2 - w_2^2)}{w_1\,\bar{w}_1}\,(w_1 - \bar{w}_1)^2 + \left[c_1\left(\frac{1}{w_1} - \frac{(w_2 + \bar{w}_2)}{\bar{w}_1}\right) + \frac{c_2 a_1}{w_2}\right](w_1 - \bar{w}_1)(w_2 - \bar{w}_2)$$

$$-c_2\,\frac{a_1\bar{w}_1}{w_2\,\bar{w}_2}\,(w_2 - \bar{w}_2)^2 - (c_3 a_4 - c_2)(1-m)w_2\,w_3 + c_2(1-m)w_3\bar{w}_2 + \frac{w_3 w_4}{a_6 + w_3}(c_4 a_7 + c_3 a_5) - c_3 a_3 w_3 - c_4 a_8 w_4$$

By choosing $c_1 = 1$, $c_2 = \frac{1}{(1-m)\,\bar{w}_2}$, $c_3 = \frac{1}{a_3}$, $c_4 = \frac{a_5}{a_3 a_7}$ we get:

$$\frac{dV_2}{dt} = \frac{-(\bar{w}_2 - w_2^2)}{w_1\,\bar{w}_1}\,(w_1 - \bar{w}_1)^2 + \left(\frac{1}{w_1} - \frac{(w_2 + \bar{w}_2)}{\bar{w}_1} + \frac{a_1}{(1-m)\,w_2\bar{w}_2}\right)(w_1 - \bar{w}_1)(w_2 - \bar{w}_2)$$

$$-\frac{a_1\bar{w}_1}{w_2\,\bar{w}_2\,\bar{w}_2{}^2}\,(w_2 - \bar{w}_2)^2 - \frac{w_2\,w_3}{\bar{w}_2} - \left[\frac{a_3 - a_4\,(1-m)\,\bar{w}_2}{a_3\,\bar{w}_2}\right] w_2\,w_3$$

$$\frac{dV_2}{dt} \le -\left[\sqrt{\frac{(\bar{w}_2 - w_2^2)}{w_1\,\bar{w}_1}}\,(w_1 - \bar{w}_1) - \sqrt{\frac{a_1\bar{w}_1}{(1-m)\,w_2\,\bar{w}_2{}^2}}\,(w_2 - \bar{w}_2)\right]^2 - \left[\frac{a_3 - a_4\,(1-m)\,\bar{w}_2}{a_3\,\bar{w}_2}\right] w_2\,w_3\ .$$

So, according to the condition (8 e) we obtain that:

$$\frac{dV_2}{dt} \le -\left[\sqrt{\frac{(\bar{w}_2 - w_2^2)}{w_1\,\bar{w}_1}}\,(w_1 - \bar{w}_1) - \sqrt{\frac{a_1\bar{w}_1}{(1-m)\,w_2\,\bar{w}_2{}^2}}\,(w_2 - \bar{w}_2)\right]^2 .$$

However, the conditions (11 a) *and* (11 b) guarantee the completeness of the quadratic term between w_1 *and* w_2 . So, if the condition (11 c) holds then we obtain that $\frac{dV_2}{dt}$ is negative definite on the region Ω_1 and hence V_2 is strictly Lyapunov function defined on the region Ω_1 . Thus E_1 is globally asymptotically stable on the region Ω_1 and the proof is complete .

Theorem (**4**): Assume that the equilibrium point $E_2 = (\breve{w}_1 , \breve{w}_2 , \breve{w}_3 , 0)$ of system (2) is locally asymptotically stable in the R_+^4 . Then E_2 is globally asymptotically stable on any region $\Omega_2 \subset R_+^4$ that satisfied the following conditions:

$$\frac{1}{w_1} - \frac{(w_2 + \breve{w}_2)}{\breve{w}_1} + \frac{a_1}{w_2} \le 2\sqrt{\frac{a_1(\breve{w}_2 - w_2^2)}{w_1\,w_2\,\breve{w}_2}} \qquad (12\ a)$$

$$\frac{w_2 + \breve{w}_2}{\breve{w}_1} < \frac{1}{w_1} + \frac{a_1}{w_2} \qquad (12\ b)$$

$$w_2^2 < \breve{w}_2 \qquad (12\ c)$$

Proof: Consider the following function:

$$V_3(w_1, w_2 , w_3 , w_4) = c_1\left(w_1 - \breve{w}_1 - \breve{w}_1\,ln\frac{w_1}{\breve{w}_1}\right) + c_2\left(w_2 - \breve{w}_2 - \breve{w}_2\ ln\frac{w_2}{\breve{w}_2}\right)$$

$$+c_3\left(w_3 - \breve{w}_3 - \breve{w}_3\,ln\frac{w_3}{\breve{w}_3}\right) + c_4 w_4\,,$$

where c_1 , c_2 , c_3 *and* c_4 are positive constants to be determine .

Clearly $V_3 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_3 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_3}{dt} = -c_1 \frac{(\breve{w}_2 - w_2^2)}{w_1 \breve{w}_1} (w_1 - \breve{w}_1)^2 + \left[c_1\left(\frac{1}{w_1} - \frac{(w_2 + \breve{w}_2)}{\breve{w}_1}\right) + \frac{c_2 a_1}{w_2}\right](w_1 - \breve{w}_1)(w_2 - \breve{w}_2)$$

$$-c_2 \frac{a_1 \breve{w}_1}{w_2 \breve{w}_2} (w_2 - \breve{w}_2)^2 + (c_3 a_4 - c_2)(1 - m)(w_2 - \breve{w}_2)(w_3 - \breve{w}_3)$$

$$+ \frac{w_3 w_4}{a_6 + w_3}(c_4 a_7 - c_3 a_5) - \left[c_4 a_8 - \frac{c_3 a_5 \breve{w}_3}{a_6 + w_3}\right] w_4$$

By choosing $c_1 = 1$, $c_2 = 1$, $c_3 = \frac{1}{a_4}$, $c_4 = \frac{a_5}{a_4 a_7}$ we get:

$$\frac{dV_3}{dt} = -\frac{(\breve{w}_2 - w_2^2)}{w_1 \breve{w}_1} (w_1 - \breve{w}_1)^2 + \left(\frac{1}{w_1} - \frac{(w_2 + \breve{w}_2)}{\breve{w}_1} + \frac{a_1}{w_2}\right)(w_1 - \breve{w}_1)(w_2 - \breve{w}_2) - \frac{a_1 \breve{w}_1}{w_2 \breve{w}_2}(w_2 - \breve{w}_2)^2 - \frac{a_5}{a_4}\left[\frac{a_8}{a_7} - \frac{\breve{w}_3}{a_6 + w_3}\right] w_4 .$$

$$\frac{dV_3}{dt} \le -\left[\sqrt{\frac{(\breve{w}_2 - w_2^2)}{w_1 \breve{w}_1}} (w_1 - \breve{w}_1) - \sqrt{\frac{a_1 \breve{w}_1}{w_2 \breve{w}_2}} (w_2 - \breve{w}_2)\right]^2 - \frac{a_5}{a_4}\left[\frac{a_8}{a_7} - \frac{\breve{w}_3}{a_6 + w_3}\right] w_4 .$$

Now, according to the condition (9 e) we obtain that:

$$\frac{dV_3}{dt} \le -\left[\sqrt{\frac{(\breve{w}_2 - w_2^2)}{w_1 \breve{w}_1}} (w_1 - \breve{w}_1) - \sqrt{\frac{a_1 \breve{w}_1}{w_2 \breve{w}_2}} (w_2 - \breve{w}_2)\right]^2$$

However, the conditions (12 a) and (12 b) guarantee the completeness of the quadratic term between w_1 and w_2 . So, if the condition (12 c) holds. Then, $\frac{dV_3}{dt}$ is negative on the region Ω_2 and hence V_3 is strictly Lyapunov function defined on the region Ω_2 . Thus E_2 is globally asymptotically stable and the proof is complet.

Theorem (**5**): Assume that the equilibrium point $E_3 = (w_1^*, w_2^*, w_3^*, w_4^*)$ of system (2) is locally asymptotically stable in the R_+^4 . Then E_3 is globally asymptotically stable on any region $\Omega_4 \subset Int\, R_+^4$ that satisfies the following conditions:

$$\frac{1}{w_1} - \frac{(w_2 + w_2^*)}{w_1^*} + \frac{a_1 a_4}{a_5 w_2} \le 2\sqrt{\frac{a_1 a_4 (w_2^* - w_2^2)}{a_5\, w_1\, w_2\, w_2^*}} \quad (13\ a)$$

$$\frac{w_2 + w_2^*}{w_1^*} < \frac{1}{w_1} + \frac{a_1 a_4}{a_5 w_2} \quad (13\ b)$$

$$w_2^2 < w_2^* \quad (13\ c)$$

$$\delta > \frac{w_4^* (w_3 - w_3^*)^2}{(a_6 + w_3)(a_6 + w_3^*)} \quad where \quad (13\ e)$$

$$\delta = \left[\sqrt{\frac{(w_2^* - w_2^2)}{w_1 w_1^*}} (w_1 - w_1^*) - \sqrt{\frac{a_1 a_4 w_1^*}{a_5 w_2 w_2^*}} (w_2 - w_2^*)\right]^2$$

Proof: Consider the following function:

$$V_4(w_1, w_2, w_3, w_4) = c_1\left(w_1 - w_1^* - w_1^* \ln \frac{w_1}{w_1^*}\right) + c_2\left(w_2 - w_2^* - w_2^* \ln \frac{w_2}{w_2^*}\right)$$

$$+c_3\left(w_3 - w_3^* - w_3^*\, ln\frac{w_3}{w_3^*}\right) + c_4\left(w_2 - w_4^* - w_4^*\; ln\frac{w_4}{w_4^*}\right),$$

where c_1 , c_2 , c_3 *and* c_4 are positive constants to be determine .

Clearly $V_4 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_4 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_4}{dt} = -c_1\,\frac{(w_2^* - w_2^2)}{w_1\, w_1^*}(w_1 - w_1^*)^2 + \left[c_1\left(\frac{1}{w_1} - \frac{(w_2 + w_2^*)}{w_1^*}\right) + \frac{c_2 a_1}{w_2}\right](w_1 - w_1^*)(w_2 - w_2^*)$$

$$-\mathrm{c}_2\,\frac{a_1 w_1^*}{\mathrm{w}_2\, w_2^*}\,(\mathrm{w}_2 - w_2^*)^2 + (c_3 a_4 - c_2)(w_2 - w_2^*)(w_3 - w_3^*)$$

$$+\frac{(c_4 a_6 a_7 - c_3 a_5)}{(a_6 + w_3^*)}\left(\frac{(w_3 - w_3^*)(w_4 - w_4^*)}{(a_6 + w_3)}\right) + \frac{c_3 a_5 w_4^*(w_3 - w_3^*)^2}{(a_6 + w_3)(a_6 + w_3^*)}$$

By choosing $c_1 = 1,\ c_2 = \frac{a_4}{a_5}\ ,\ c_3 = \frac{1}{a_5}\ ,\ c_4 = \frac{a_6 + w_3^*}{a_6 a_7}$ we get:

$$\frac{dV_3}{dt} \le -\left[\sqrt{\frac{(w_2^* - w_2^2)}{w_1\, w_1^*}}\,(w_1 - w_1^*) - \sqrt{\frac{a_1 a_4 w_1^*}{a_5 w_2 w_2^*}}\,(w_2 - w_2^*)\right]^2 + \frac{w_4^*(w_3 - w_3^*)^2}{(a_6 + w_3)(a_6 + w_3^*)}$$

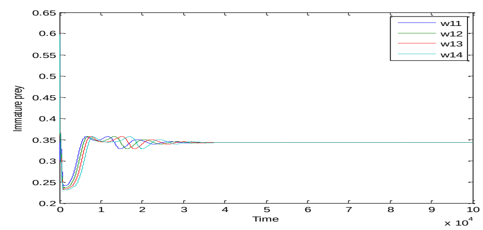
However, the conditions (13 a) *and* (13 b) guarantee the completeness of the quadratic term between w_1 *and* w_2 . So, if the condition (13 c) *and* (13 e) holds then we obtain that $\frac{dV_4}{dt}$ is negative definite on the region Ω_3 and hence V_3 is strictly Lyapunov function defined on the region Ω_3 . Thus E_3 is globally asymptotically stable on the region Ω_3 and the proof is complete .

VI. NUMERICAL ANALYSIS OF SYSTEM

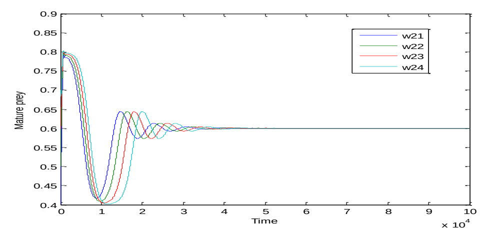
In this section, the dynamical behavior of system (2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in Fig. (1) .

$a_1 = 0.7\ , a_2 = 0.2\ , a_3 = 0.1\ , a_4 = 0.5\ , a_5 = 0.6$

$a_6 = 0.4\ , a_7 = 0.5\ , a_8 = 0.2\ , m =$



0.8 (14)



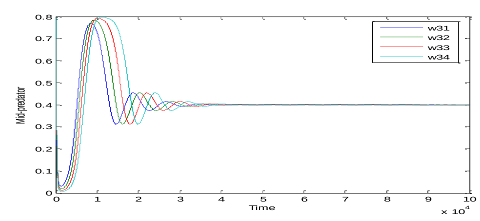


Fig.* (1): *Time series of the solution of system* (2) *that started from four different initial points
(0.3 , 0.4 , 0.5 , 0.6), (2.5, 0.5 , 2.5 , 1.5), (2 , 1.5 , 2.5, 2.5)

and $(2.5\,,2.5,1.5\,,0.5)$ ***for the data given by*** $(\,2.1\,)$. (a) ***trajectories of*** w_1 ***as a function of time,*** (b) ***trajectories of*** w_2 ***as a function of time,*** (c) ***trajectories of*** w_3 ***as a function of time,*** (d) ***trajectories of*** w_4 ***as a function of time .***

Clearly, Fig. (1) shows that system (2) has a globally asymptotically stable as the solution of system (2) approaches asymptotically to the positive equilibrium point $E_3 = (\,0.34\,,0.6\,,0.4\,,0.08\,)$ starting from four different initial points and this is confirming our obtained analytical results.

Now, in order to discuss the effect of the parameters values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in (14) with varying one parameter at each time. It is observed that for the data given in (14) with $0.1 \le a_1 < 1$, the solution of system (2) approaches asymptotically to the positive equilibrium point as shown in Fig. (2) for typical value $a_1 = 0.2$.

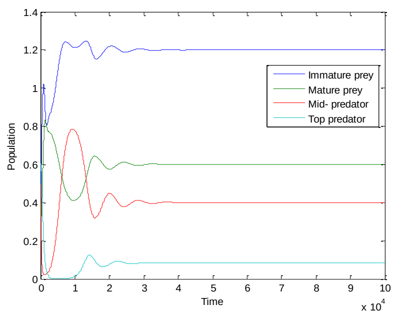


Fig. **(2)*****: Time series of the solution of system*** **(2)** ***for the data given by*** **(14)** ***with*** $a_1 = 0.2$ ***, which approaches to*** $(\,1.2\,,0.6\,,0.4\,,0.08\,)$ ***in the interior of*** R_+^4 ***.***

By varying the parameter a_2 and keeping the rest of parameters values as in (14), it is observed that for $0.1 \le a_2 < 0.44$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_3 , while for $0.44 \le a_2 < 0.6$ the solution of system (2) approaches asymptotically to $E_2 = (\,\breve{w}_1\,,\breve{w}_2\,,\breve{w}_3\,,0\,)$ in the interior of the positive quadrant of $w_1w_2w_3 - plane$ as shownin Fig.(3) for typical value $a_2 = 0.5$.

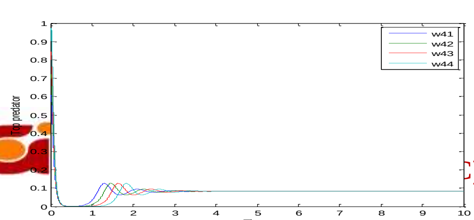
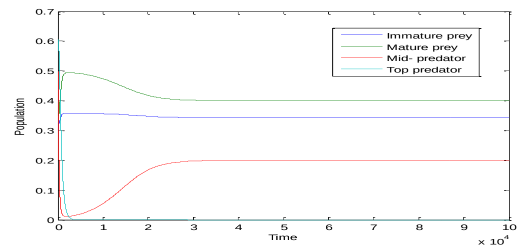


Fig. **(3)**: ***Times series of the solution of system*** **(2)** ***for the data given by*** **(14)** ***with*** $a_2 = 0.5$. ***which approaches to*** $(\,0.34\,,0.4\,,0.2\,,0\,)$ ***in the interior of the positive quadrant of*** $w_1w_2w_3 - plane$ ***.***

while for $0.6 \le a_2 < 1$ the solution of system (2) approaches asymptotically to $E_1 =$



$(\,\bar{w}_1\,,\bar{w}_2\,,0\,,0\,)$ in the interior of the positive quadrant of $w_1w_2 - plane$ as shown in Fig. (4) for typical value $a_2 = 0.8$.

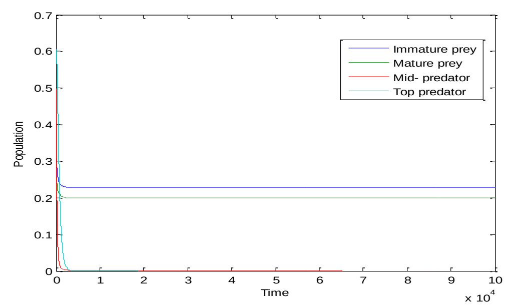


Fig. **(4)**: ***Times series of the solution of system*** **(2)** ***for the data given by*** **(14)** ***with*** $a_2 = 0.8$ ***which approaches to*** $(\,0.23\,,0.2\,,0\,,0\,)$ ***in the interior of the positive quadrant of*** $w_1w_2 - plane$ ***.***

On the other hand varying the parameter a_3 and keeping the rest of parameters values as in (14), it is observed that for $0.1 \le a_3 < 0.145$ the solution of system (2) approaches asymptotically to the positive equilibrium point E_3. while for $0.145 \le a_3 < 0.2$ the solution of system (2) approaches asymptotically to $E_2 = (\,\breve{w}_1\,,\breve{w}_2\,,\breve{w}_3\,,0\,)$in the interior of the positive quadrant of $w_1w_2w_3 - plane$. for $0.2 \le a_3 < 1$ the solution of system (2) approaches asymptotically to $E_1 = (\,\bar{w}_1\,,\bar{w}_2\,,0\,,0\,)$ in the interior of the positive quadrant of $w_1w_2 - plane$.

Moreover, varying the parameter a_4 and keeping the rest of parameters values as in (14), it is observed that for $0.1 \leq a_4 < 0.26$ the solution of system (2) approaches asymptotically to $E_1 = (\overline{w}_1, \overline{w}_2, 0, 0)$ in the interior of the positive quadrant of $w_1w_2 - plane$,while for $0.26 < a_4 \leq 0.4$ the solution of system (2) approaches asymptotically to $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ in the interior of the positive quadrant of $w_1w_2w_3 - plan$.and for the $0.4 < a_4 < 1$ the solution of system (2) approaches asymptotically to the positive equilibrium point E_3.

For the parameter $0.5 < a_5 \leq 1$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_3.

For the parameter $0.1 \leq a_6 < 0.2$ the solution of system (2) approaches asymptotically to $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ in the interior of the positive quadrant of $w_1w_2w_3 - plan$. While for the $0.2 \leq a_6 < 0.5$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_3.

For the parameters values given in (14) with $0.4 < a_7 < 0.6$ the solution of system (2) approaches asymptotically to the positive equilibrium point E_3.

For the parameter $0.1 \leq a_8 < 0.32$ the solution of system (2) approaches asymptotically to the positive equilibrium point E_3.while for the $0.32 \leq a_8 < 1$ the solution of system (2) approaches asymptotically to $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ in the interior of the positive quadrant of $w_1w_2w_3 - plan$.

Moreover, varying the parameter m and keeping the rest of parameters values as in (14), it is observed that for $0.1 \leq m < 0.7$ the solution of system (2) approaches asymptotically to the positive equilibrium point E_3. while for $0.7 \leq m < 0.75$ the solution of system (2) approaches asymptotically to $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$in the interior of the positive quadrant of $w_1w_2w_3 -$ plane. and for $0.75 \leq m < 1$

the solution of system (2) approaches asymptotically to $E_1 = (\overline{w}_1, \overline{w}_2, 0, 0)$ in the interior of the positive quadrant of $w_1w_2 -$ plane.

Finally, the dynamical behavior at the vanishing equilibrium point $E_0 = (0, 0, 0, 0)$ is investigated by choosing $a_2 = 2$ and keeping other parameters fixed as given in (14), and then the solution of system (2) is drawn in Fig. (5) .

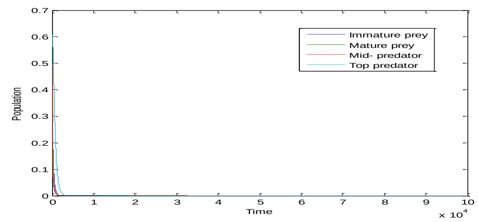


***Fig.*(5): *Times series of the solution of system* (2) *for the data given by* (14) *of with* $a_2 = 2$ *, which* approaches (0 , 0 , 0 , 0).**

VII. CONCLUSIONS AND DISCUSSION

In the paper chapters, we proposed and analyzed a prey-predator system incorporating a stage structure of prey with refuge. It is assumed that the mid- predator species prey upon the mature prey according to Lotka-Voltera type of functional response and assumed that the top predator species upon the mid- predator according to Holling type-II functional response . The existence, uniqueness and bounded ness of the solution of the system are discussed. The existence of all possible equilibrium points is studied. The local and global dynamical behaviors of the system are studied analytically. Finally, to understand the effect of varying each parameter on the global dynamics of system (2) has been solved numerically for a biological feasible set of hypothetical parameters values and the following results are obtained:

1. System (2) has only one type of attractor in Int. R^4_+ approaches to globally stable point.

2. For the set hypothetical parameters value given in (14), the system (2) approaches asymptotically to globally stable positive point $E_3 = (0.34, 0.6, 0.4, 0.08)$.Further, with varying one parameter each time, it is observed that varying the parameter values, $a_{i}, i = 1,5$ *and* 7 do not have any effect on the dynamical behavior of system (2) and the solution of the system still approaches to positive equilibrium point $E_3 = (w_1^*, w_2^*, w_3^*, w_4^*)$.

3. As the natural death rate of mature prey a_2 increasing to 0.43 keeping the rest of parameters as in Eq.(14),the solution of system (2) approaches to positive equilibrium point E_3. However if $o.44 \leq a_6 < 0.6$,then the top predator will face extinction and the solution of system (2) approaches asymptotically to $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$,moreover ,increasing $a_2 \geq 0.6$ will causes extinction in the mid-predator and top predator and the solution of system (2) approaches asymptotically to $E_1 = (\bar{w}_1, \bar{w}_2, 0, 0)$.Further ,it is observed that the natural death rate of mid-predator a_3 and the number of prey inside the refuge m have the same effect as a_2 .

5. As the half saturation rate of top predator a_6 decreases keeping the rest parameters as in eq (14) the top predator will face extinction and the solution of system (2) approaches to$E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$. but where a_6 increases system (2) still has a globally asymptotically stable positive point in the int. R^4_+.

6. As the natural death rate of the top predator $a_8 \geq 0.1$ increasing keeping the rest of parameters as in Eq (14) has asymptotically stable positive point in Int. R^4_+.However increasing $a_8 \geq 0.32$ will causes extinction in the top predator and the solution of system (2) approaches asymptotically to $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$.

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